Free Boolean extensions of Heyting algebras

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Abstract

Free Boolean extension of Heyting algebras play a crucial role in the algebraic translation of intuitionistic logic into modal S4 logic, in particular in the proof of the Blok-Esakia theorem. Based on his duality, Esakia provided a characterization of free Boolean extensions. Algebraically, is may be expressed with the use of stable homomorphism and simple interior algebras. W provide a direct proof of this fact.

1 Introduction

It is well known that every bounded distributive lattice $\mathbf{L}$ embeds into some Boolean lattice. Actually, one can choose a "minimal" Boolean lattice into which $\mathbf{L}$ embeds. It is called the *free Boolean extension of* $\mathbf{L}$. Indeed, if $\mathbf{L}$ embeds into Boolean lattices $\mathbf{A}$, $\mathbf{B}$ and also $\mathbf{A}$, $\mathbf{B}$ are generated as Boolean algebras by the carrier $\mathbf{L}$, then $\mathbf{A}$ and $\mathbf{B}$ are isomorphic [1, Section V.4].

This fact is crucial in algebraic studies of the relationship of intuitionistic logic and modal logic S4. Stictly, the connection of Heyting algebra with interior algebra is given by the following McKinsey-Tarski theorem [11, Section 1] (see also [4, Theorem 2.2] and [10, Section 3]). Recall that open elements of an interior algebra $\mathbf{M}$ form the Heyting algebra $\mathsf{O}(\mathbf{M})$ with the order structure inherited from $\mathbf{M}$.

Theorem 1. *For every Heyting algebra* $\mathbf{H}$ *there is an interior algebra* $\mathsf{B}(\mathbf{H})$ *such that*

1. $\mathsf{OB}(\mathbf{H}) = \mathbf{H}$*;*
2. *for every interior algebra* $\mathbf{M}$*, if* $\mathbf{H} \leqslant \mathsf{O}(\mathbf{M})$*, then* $\mathsf{B}(\mathbf{H})$ *is isomorphic to the subalgebra of* $\mathbf{M}$ *generated by* H*;*

The algebra $\mathsf{B}(\mathbf{H})$ is called the *free Boolean extension of* $\mathbf{H}$. From Theorem 1 it follows directly that the class of free Boolean extensions of Heyting algebras coincides (up to isomorphism of algebras) with the class of interior algebras generated by open elements.

The Blok-Esakia theorem states that the intermediate logics are in one to one correspondence with the normal extension of Grzegorczyk modal logic. The algebraic proof of this fact is based on the understanding of the structure of Grzegorczyk modal algebras, i.e., algebras from the variety generated by free Boolean extensions of Heyting algebras [3, 12].

L. Esakia developed duality theory for interior algebras [5, 6]. In particular, he provided a characterization of free Boolean extensions of Heyting algebras. It says that an interior algebra is isomorphic to a free Boolean extension of a Heyting algebras if and olny if its dual descriptive frame has no non-trivial clusters [6, Theorem 12.7]. An algebraic terms it may be formulated as follows (see the next section for the definitions).

Theorem 2. *An interior algebra is isomorphic to a free Boolean extension of a Heyting algebra if and only if it does not admits a stable homomorphism onto a four-element simple interior algebra* $\mathbf{S}_2$.

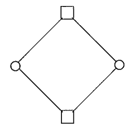
We provide an algebraic direct proof of this theorem.

2 Proof

A modal algebra $\mathbf{M}$ is called an *interior algebra* if for every $a \in M$ it satisfies

$$a \leqslant \Box a = \Box\Box a.$$

An element a of $\mathbf{M}$ is *open in* $\mathbf{M}$ if $\Box a = a$. Recall that for a modal algebra there is one to one correspondence between its congruences and its open filters, i.e., Boolean filters closed under $\Box$ operation. In particular for an interior algebra, an element b belongs to the open filter generated by a iff $\Box a \leqslant b$. It follows that an interior algebra is simple iff it has exactly two open elements 0 and 1. A four-element simple algebra, denoted by $\mathbf{S}_2$ is depicted bellow (open elements are marked by $\Box$).



Let $\mathbf{M}$ and $\mathbf{N}$ be modal algebras. We say that a mapping $f\colon N \to M$ is a stable homomorphism from $\mathbf{N}$ into $\mathbf{M}$ if it is a Boolean homomorphism and

$$f(\Box a) \leqslant \Box f(a)$$

holds for every $a \in M$. The reader may consult e.g. [2, 7] for the importance of stable homomorphisms in modal logic (in [7] they are called *continuous morphism*). Note that a stable homomorphism does not need to be a homomorphism. For instance, $\mathbf{S}_2$ does not admit a homomorphism onto a two-element interior algebra, but it admits such stable homomorphisms.

We need the following known extensions of prime filter theorem for Boolean algebras.

Lemma 3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be Boolean algebras,* $\mathbf{D}$ *be a subalgebra of* $\mathbf{C}$ *and* U *be an ultrafilter of* $\mathbf{D}$ *Then there exists an ultrafilter* W *of* $\mathbf{C}$ *such that* $W \cap D = U$.

Proof. Let $F = \{c \in C \mid (\exists d \in U)\ d \leqslant c\}$. Then F is a proper filter of $\mathbf{C}$. By the prime filter theorem, there exists an ultrafilter W of $\mathbf{C}$ extending F. Then $W \cap D$ is an ultrafilter of $\mathbf{D}$ extending U. By the maximality of U (as a proper filter of $\mathbf{D}$), we have $W \cap D = U$. □

The following less obvious lemma may be found, as a harder exercise, in [8, Exercise 15 in Chapter 20]. Its proof may be found in [9]. We provide also a new, and more direct proof of this fact. Actually, the formulation in [9] is slightly stronger. Our proof yields also this extension.

Lemma 4. *Let* $\mathbf{B}$ *be a proper Boolean subalgebra of* $\mathbf{A}$*. Then there are two distinct ultrafilters* U_1 *and* U_2 *of* $\mathbf{A}$ *such that* $U_1 \cap B = U_2 \cap B$.

Proof. It is convenient to formulate this prove with the use of homomorphisms. Recall that ultrafilters of Boolean algebras are exactly preimages of 1 for homomorphisms onto a two-element Boolean algebra $\mathbf{2}$.

Let $e \in A - B$. Let G_1 and G_2 be two different proper filters of $\mathbf{A}$ such that $e \in G_1$, $\neg e \in G_2$ and

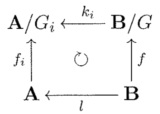
$$G_1 \cap B = G_2 \cap B = G.$$

For instance, we may take

$$G_1 = \{a_1 \wedge a_2 \in A \mid e \leqslant a_1 \text{ and } (\exists b \in B)\, \neg e < b \leqslant a_2\}$$
$$G_2 = \{a_1 \wedge a_2 \in A \mid (\exists b \in B)\, e < b \leqslant a_1 \text{ and } \neg e \leqslant a_2\}.$$

Since e and $\neg e$ do not belong to B and B is closed under $\wedge$ operation, such defined G_1 and G_2 are indeed filters of $\mathbf{A}$. Moreover, $e \in G_1$, $\neg e \in G_2$. Furthermore, if $a_1 \geqslant e$ and $a_2 > \neg e$, then $a_1 \wedge a_2 > 0$. This shows that G_1 is proper. Similarly we infer that G_2 is proper. Also it is the case that $G_1 \cap B = G_2 \cap B$.

Let $l\colon \mathbf{B} \to \mathbf{A}$ be the embedding. Let us also define $f_i\colon \mathbf{A} \to \mathbf{A}/G_i$; $a \mapsto a/G_i$, and $f\colon \mathbf{B} \to \mathbf{B}/G$; $b \to b/G$. Since $G_i \cap B = G$, the homomorphisms $k_i\colon \mathbf{B}/G \to \mathbf{A}/\mathbf{G}_i$; $b/G \to b/G_i$ are well defined and injective. We have the following commutative diagram.



By the prime filter theorem, there exists a homomorphism $g\colon \mathbf{B}/G \to \mathbf{2}$. By Lemma 3, there are surjective homomorphisms $g_i\colon \mathbf{A}/G_i \to \mathbf{2}$ such that $g = g_i \circ k_i$. Let

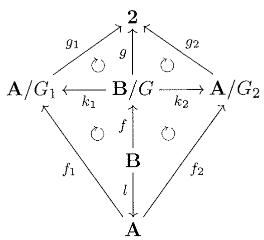
$$U_i = (g_i \circ f_i)^{-1}(1).$$

Since $G_i \subseteq U_i$, we have $e \in U_1$ and $\neg e \in U_2$. This yields that $U_1 \neq U_2$. Furthermore, we have

$$(g \circ f)^{-1}(1) = (g_i \circ k_i \circ f)^{-1}(1) = (g_i \circ f_i \circ l)^{-1}(1) = U_i \cap B.$$

Thus $U_1 \cap B = U_2 \cap B$.

The proof is illustrated by the following "kite" diagram (the commuting parts are marked by ↻).



□

Lemma 5. *Let* $\mathbf{M}$ *be an interior algebra and* $\mathbf{S}$ *be a simple interior algebra. Then the mapping* $f\colon M \to S$ *is a stable homomorphism if and only if it is a Boolean homomorphism and for every* $a \in M$ *we have* $f(\Box a) \in \{0,1\}$.

Proof. Assume that f is a stable homomorphism. By the definition, f is a Boolean homomorphism. Moreover, if $f(\Box a) < 1$ then $f(\Box a) = f(\Box\Box a) \leqslant \Box f(\Box a) = 0$. For the oposie implication assume that f is a Boolean homomorphism and $f(\Box a) = \{0,1\}$. If $f(\Box a) = 0$ then, clearly, $f(\Box a) \leqslant \Box f(a)$. If $f(\Box a) = 1$ then, since $\Box a \leqslant a$, $f(a) = 1$, and so $f(\Box a) = 1 = \Box f(a)$. □

Proof of Theorem 2. Let $\mathbf{M}$ be an interior algebra and $\mathbf{N}$ be its subalgebra generated by all open elements of $\mathbf{M}$.

Assume first that $\mathbf{N}$ is a proper subalgebra of $\mathbf{M}$. Let $\mathbf{A}$ and $\mathbf{B}$ be Boolean reducts of $\mathbf{M}$ and $\mathbf{N}$ respectively. Let U_1 and U_2 be ultrafilters from Lemma 4. Let $\mathbf{S}$ be the simple interior algebras (only with 0 and 1 open) with the Boolean reduct $\mathbf{A}/U_1 \times \mathbf{A}/U_2$. Then $\mathbf{S}$ is isomorphic to $\mathbf{S}_2$. Let $f\colon A \to S$ be given by $a \mapsto (a/U_1, a/U_2)$. Then f is a surjective Boolean homomorphism. Moreover, since $U_1 \cap N = U_2 \cap N$, $f(N) = \{0,1\}$. Thus, by Lemma 5, f is a stable homomorphism from $\mathbf{M}$ onto $\mathbf{S}$.

Now assume that $\mathbf{M}$ is generated by its open elements. Then $\mathbf{M}$ is also Boolean generated by its open elements. Let f be a stable homomorphism from $\mathbf{M}$ into $\mathbf{S}_2$. By Lemma 5, for every $a \in M$ we have $f(\Box a) \in \{0,1\}$. Since f preserves Boolean operations and $\{0,1\}$ is closed under Boolean operations, f maps M onto $\{0,1\}$. Thus f is not surjective.

□

Example 6. Let us consider the following algebra $\mathbf{M}$. The carrier M is the power set of the set $\mathbb{N}$ of natural numbers. The Boolean operations are the set theoretic operations (in particular, $1 = \mathbb{N}$, and $0 = \emptyset$). Let

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ \{0,\ldots,k-1\} & \text{if } a \neq 1 \text{ and } k = \min \neg a \end{cases}.$$

Note that $\mathbf{M}$ is the dual algebra for the modal frame $(\mathbb{N}, \geqslant)$ in the Jónsson-Tarski duality. It is known that $\mathbf{M}$ is a Grzegorczyk algebra, i.e., it belongs to the variety generated by free Boolean extensions of Heyting algebras.

Let F be the Boolean filter of all cofinite subsets of $\mathbb{N}$. Let $\mathbf{S}$ be the simple interior algebra with the Boolean reduct obtained by dividing the Boolean reduct of $\mathbf{M}$ by F, and only two open elements 0 and 1. Then $f\colon a \to a/F$ is a stable homomorphism from $\mathbf{M}$ onto $\mathbf{S}$. Moreover, since $\mathbf{S}$ is infinite, it admits a stable homomorphism onto $\mathbf{S}_2$. Thus $\mathbf{M}$ admits a stable homomorphism onto $\mathbf{S}_2$.

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